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Characterization of Strict Approximations in Subspaces of Spline Functions

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The problem of approximating a given function by spline functions with fixed knots is discussed. Strict approximations which are particular unique best Chebyshev approximations are considered. The chief purpose is to develop a characterization theorem for these strict approximations.

INTRODUCTION

In this paper we consider the problem of approximating a given function f in $C(T)$ by spline functions with fixed knots. One of the difficulties lies in the fact that a best approximation is not always unique. Therefore it is natural to consider conditions which single out one of the best approximations. Rice [3] defines a unique "strict approximation" for functions defined on a finite set. But discrete approximation problems are closely related to problems on an interval. Recently strict approximations were extensively studied. It is possible to determine these approximations by methods known as ascent methods.

The chief purpose of this paper is to develop characterization theorems for strict approximations in subspaces of spline functions.

Rice [3] and Schumaker [4] established characterization theorems for best approximations to functions f defined on an interval.

In Section 2 we shall extend these results to the problem where T is a compact subset of $\mathbb{R}$ and the subspaces of spline functions satisfy certain boundary conditions.

These results will be used in Section 3 in order to characterize strict approximations. First we single out a uniquely determined function in the set of best approximations by an inductive definition. Then we shall show that this best approximation is the strict approximation in the sense of Rice. Using our definition it is possible to establish a characterization theorem for

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strict approximations (Theorem 3.10). Moreover our definition can be applied to problems defined on an interval.

In Section 4 we shall consider such approximation problems and shall show the difficulties which arise from these problems. Then we define for a great class of continuous functions uniquely determined best approximations which can also be considered as strict approximations.

The characterization theorem is very useful for developing an algorithm which determines the strict approximation. In a further paper we shall establish such an algorithm.

1. Preliminaries

Let T be a compact subset of $\mathbb{R}$, let $C(T)$ be the normed linear space of all continuous real-valued functions defined on T and let the space $C(T)$ be normed by

$$\|f\| = \max_{x \in T} |f(x)|.$$

Suppose f is a function in $C(T)$ and $G = \operatorname{span}\{g_1, \dots, g_n\}$ an n-dimensional subspace of $C(T)$. Then we consider the linear Chebyshev approximation problem: For every function f in $C(T)$ we define the set of *best approximations* to f out of G by

$$P_G(f) := \{g_0 \in G: \|f - g_0\| = \inf\{\|f - g\|: g \in G\}\}.$$

We shall use the notations

$$A\begin{pmatrix} t_1, \dots, t_h \\ g_1, \dots, g_n \end{pmatrix} = \begin{pmatrix} g_1(t_1) & \cdots & g_n(t_1) \\ \vdots & & \vdots \\ g_1(t_h) & \cdots & g_n(t_h) \end{pmatrix}$$

where $\{t_i\}_{i=1}^{h}$ is a subset of T, we denote by $E(f)$ the set of *extremal points* of the function f (on T)

$$E(f) = \{x \in T: |f(x)| = \|f\|\}$$

and we call points $t_1 < \cdots < t_h$ in T *alternating extremal points* of f if $\eta(-1)^i f(t_i) = \|f\|$, $i = 1, \dots, h$, $\eta \in \{-1, 1\}$.

The following theorem is well-known (see Watson [7, p. 50]).

THEOREM 1.1. *A function g_0 is an element of $P_G(f)$ if and only if there exists $E \subset E(f - g_0)$ containing $h \leqslant n + 1$ points $t_1, ..., t_h$ and a nontrivial vector $\lambda \in \mathbb{R}_h$ such that*

$$\lambda^T A \begin{pmatrix} t_1 & \cdots & t_h \\ g_1 & \cdots & g_n \end{pmatrix} = 0,$$

$$\lambda_i \cdot \sigma_i \geqslant 0, \qquad i = 1, ..., h$$

where $\sigma_i = \operatorname{sign}((f - g_0)(t_i))$, $i = 1, ..., h$.

In later sections we shall also consider approximation problems on a subset U of T. A function g_0 in G is called a *best approximation* to a function f on U (out of G) iff

$$\|(f - g_0)|_U\| = \inf_{g \in G} \{\max_{x \in U} |f(x) - g(x)|\}.$$

A subset $U = \{u_1, ..., u_{n+1}\}$ of T is called a *reference* iff

$$A_U = A \begin{pmatrix} u_1 & \cdots & u_{n+1} \\ g_1 & \cdots & g_n \end{pmatrix}$$

has rank n.

Suppose that the subset $U = \{u_1, ..., u_{n+1}\}$ in T is a reference. Then there exists a unique solution (up to a scalar) of the linear system $\lambda^T A_U = 0$, where $\lambda^T = (\lambda_1, ..., \lambda_{n+1})$ is a nontrivial vector in $\mathbb{R}_{n+1}$. We obtain

$$\lambda_i = c \cdot (-1)^i \det A \begin{pmatrix} u_1 & \cdots & u_{i-1} u_{i+1} & \cdots & u_{n+1} \\ g_1 & & \cdots & & g_n \end{pmatrix}, \qquad c \in \mathbb{R}. \quad (1.1)$$

Let g_0 be a function of G satisfying

$$\begin{aligned} f(u_i) - g_0(u_i) &= \eta \sigma_i \|(f - g_0)|_U\|, \\ \sigma_i &= \lambda_i / |\lambda_i| \qquad \text{for all } \lambda_i \neq 0 \\ \sigma_i &\in \{-1, 1\} \qquad \text{elsewhere} \end{aligned} \quad (1.2)$$

where λ_i is defined in (1.1). Then it follows from Theorem 1.1 that g_0 is a best approximation to f on the reference U. We call $\gamma = \|(f - g_0)|_U\|$ the *reference deviation.*

A subspace G satisfies the *Haar condition* if $g \in G$, $g(x) = 0$ at n distinct points of T implies $g \equiv 0$. In this case the best approximation is always unique.

2. Characterization Theorems

In this section we shall study approximation problems for subspaces of polynomial spline functions. Let Δ denote the partition $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ on the interval $[a, b]$. The subspace $S_m(\Delta)$ of polynomial spline functions of degree m ($m \geqslant 2$) with simple fixed knots at Δ is defined by

$$S_m(\Delta) = \{s \in C^{m-2}[a, b]: s|_{[x_i, x_{i+1}]} \in \Pi_{m-1}, i = 0,..., k\}$$

where Π_{m-1} denotes all polynomials of degree $\leqslant m - 1$.

Let be $T = [a, b]$ and $G = S_m(\Delta)$ in the approximation problem of Section 1. Rice [3] and Schumaker [4] established characterization theorems for the best approximations of this problem.

In our investigations it will be necessary to study approximation problems defined an a compact subset T of $[x_0, x_{k+1}]$. Moreover, we consider subspaces of $S_m(\Delta)$ satisfying boundary conditions. Therefore we shall extend the results of Rice and Schumaker to these problems. The following notation is used throughout the paper: K_1 is the class of closed subintervals, K_2 and K_3 are the classes of half-open subintervals of the form $(u, v]$ and $[u, v)$, respectively; K_4 is the class of open subintervals. We shall denote by $I_{p,q}$ a subinterval with boundary points x_p and x_q where x_p and x_q are knots of the subspace of spline functions. Then $I_{p,q} = [x_p, x_q]$ if $I_{p,q} \in K_1$, $I_{p,q} = (x_p, x_q]$ if $I_{p,q} \in K_2$, $I_{p,q} = [x_p, x_q)$ if $I_{p,q} \in K_3$ and $I_{p,q} = (x_p, x_q)$ if $I_{p,q} \in K_4$.

Using these notations we shall define the following subspaces: Let T be a compact subset of $[x_0, x_{k+1}]$ and let I be a subinterval such that $(x_0, x_{k+1}) \subset I \subset [x_0, x_{k+1}]$. Then

$$\begin{aligned} S_m(I, T) = \{s|_T: s \in S_m(\Delta), & \\ & \text{if } I \in K_2 \text{ then } s^{(i)}(x_0) = 0,\ i = 0,..., m-2, \\ & \text{if } I \in K_3 \text{ then } s^{(i)}(x_{k+1}) = 0,\ i = 0,..., m-2, \\ & \text{if } I \in K_4 \text{ then } s^{(i)}(x_0) = s^{(i)}(x_{k+1}) = 0,\ i = 0,..., m-2\}. \end{aligned}$$

If $T = [x_0, x_{k+1}]$ then we denote $S_m(I, T)$ by $S_m(I)$. Notice that $S_m(I, T)$ satisfies boundary conditions in x_0 or x_{k+1} if the interval I is not closed. If $I \in K_1$ then $S_m(I) = S_m(\Delta)$. We see that the interval I also determines the boundary conditions of the subspace in consideration.

These notations will be very useful in Section 3.

Remark 2.1. Let $\tilde{\Delta}$ denote the partition $x_{-m+1} < \cdots < x_0 < \cdots < x_{k+1} < \cdots < x_n$, where $n = m + k$. Let $\tilde{I} = (x_{-m+1}, x_n)$, $I = [x_0, x_{k+1}]$ and $f \in C(T)$, where T is a closed subset of I. Then $S_m(\tilde{I}, T) = S_m(I, T)$. This is also true if $I=(x_{-m+1}, x_{k+1}]$ or $I=[x_0, x_n)$. Therefore it is sufficient to establish characterization theorems for best approximations only for the case $G=S_m(\tilde{I}, T)$. The other cases will immediately follow from these results.

A local basis of $S_m(\tilde{I})$ will be very useful. Let M_i be the mth order B-spline associated with the knots $x_i,..., x_{i+m}$ (see [5, p. 118]). Then

$$S_m(\tilde{I})=\operatorname{span}\{M_{-m+1},..., M_{n-m}\}.$$

If we consider $S_m(\tilde{I}, T)$ then we also denote $M_i|_T$ by M_i.

Let T be a closed subset of $\tilde{I}$ and $U=\{u_i\}_{i=1}^n \subset T$. Then

$$A\begin{pmatrix} u_1 & \cdots & u_n \\ M_{-m+1} & \cdots & M_k \end{pmatrix}$$

has rank n if and only if

$$x_{-m+i}<u_i<x_i, \qquad i=1,..., n \tag{2.1}$$

(see [5]). Therefore the dimension of $S_m(\tilde{I}, T)$ is n, iff there exists a subset $U \subset T$ satisfying (2.1).

Now we want to prove the following lemma.

LEMMA 2.2. *Let the partition* $\tilde{\Delta}=\{x_i\}_{i=-m+1}^n$, $m \geqslant 2$ *and* $n \geqslant 1$, *be given, let* $\tilde{I}=(x_{-m+1}, x_n)$ *and let* $T=\{u_i\}_{i=1}^{n+1}$ *be a subset of* $\tilde{I}$ *where* $x_{-m+1}<u_1<\cdots<u_{n+1}<x_n$.

(a) *Then there exist positive integers* p *and* q, $1 \leqslant p \leqslant q \leqslant n$, *and a subset* $\{v_i\}_{i=p}^{q+1} \subset T \cap (x_{-m+p}, x_q)$ *such that*

$$v_i \in (x_{-m+i}, x_{i-1}), \qquad i=p+1,..., q. \tag{2.2}$$

If $p>1$ *then* $v_p \geqslant x_{p-1}$ *and if* $q<n$ *then* $v_{q+1} \leqslant x_{-m+q+1}$.

(b) *Let the dimension of* $S_m(\tilde{I}, T)$ *be* n. *Then there exist positive integers* p *and* q, $1 \leqslant p \leqslant q \leqslant n$, *such that* $\{u_i\}_{i=p}^{q+1} \subset (x_{-m+p}, x_q)$ *and*

$$\begin{aligned} u_i &\in (x_{-m+i}, x_{i-1}), & i&=p+1,..., q, \\ u_i &\in (x_{-m+i}, x_i), & i&=1,..., p, \\ u_{i+1} &\in (x_{-m+i}, x_i), & i&=q,..., n. \end{aligned} \tag{2.3}$$

If $p>1$ *then* $u_p \geqslant x_{p-1}$ *and if* $q<n$ *then* $u_{q+1} \leqslant x_{-m+q+1}$.

Proof. We shall only prove (b). The proof proceeds by induction on k. The case $n=1$ can be easily shown. Set $p=q=1$ then (2.3) is satisfied. We assume that the result has been established for $n-1$. Now we show the assertion for n. It is necessary to distinguish the following cases:

(i) Suppose that $u_i \in (x_{-m+i}, x_{i-1})$, $i = 2,\dots, n$. Then $p = 1$ and $q = n$ are the desired positive integers.

(ii) Let $u_{h+1} \leqslant x_{-m+h+1}$ for some h where $1 \leqslant h \leqslant n-1$. Then we apply the induction hypothesis to the subset $\{u_1,\dots, u_{h+1}\}$ and the subinterval (x_{-m+1}, x_h). Hence there exist p and q, $1 \leqslant p \leqslant q \leqslant h$, such that $\{u_i\}_{i=1}^{h+1}$ satisfies (2.3) on (x_{-m+1}, x_h). If $q = h$ then $u_{h+1} \leqslant x_{-m+h+1}$ and if $q < h$ then it follows from the induction hypothesis that $u_{q+1} \leqslant x_{-m+q+1}$. Moreover, we conclude from (2.1) that $u_{i+1} \in (x_{-m+i}, x_i)$, $i = q,\dots, n$. Therefore the integers p and q also satisfy (2.3) for T on $\tilde{I}$.

(iii) Let $u_h \geqslant x_{h-1}$ for some h where $2 \leqslant h \leqslant n$. This case an be similarly proven.

The assertion (a) can be similarly shown.

According to this lemma we shall define subsets which are associated with subintervals. This notation will be very important for our approximation problems.

Let $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$ be given, let $\tilde{I} = (x_{-m+1}, x_n)$ and $R = \{u_i\}_{i=p}^{q+1}$, $1 \leqslant p \leqslant q \leqslant n$, be a subset of $\tilde{I}$. We call the subset R *associated with a subinterval* J_R, iff $R \subset J_R$ where

$$J_R = \begin{cases} (x_{-m+1}, x_n) & p = 1, q = n, \\ [x_{p-1}, x_n) & p > 1, q = n, \\ (x_{-m+1}, x_{-m+q+1}] & \text{if} \quad p = 1, q < n, \\ [x_{p-1}, x_{-m+q+1}] & p > 1, q < n. \end{cases} \tag{2.4}$$

$$u_i \in (x_{-m+i}, x_{i-1}), \qquad i = p+1,\dots, q.$$

First we obtain the following result:

THEOREM 2.3. *Let the partition* $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$ *be given, let* $\tilde{I} = (x_{-m+1}, x_n)$ *and let* $R = \{u_i\}_{i=p}^{q+1}$ *be a subset of* $\tilde{I}$ *which is associated with a subinterval* J_R. *Then the following assertions will hold*:

(a) *The subspace* $S_m(\tilde{I}, R)$ *satisfies the Haar condition.*

(b) *There exists a reference* $R_1 = \{u_i\}_{i=1}^{n+1}$ *in* $\tilde{I}$ *satisfying* $R \subset R_1$ *and*

$$\det(A_i) = \begin{cases} \neq 0, & \text{if} \quad u_i \in R, \\ 0 & \text{elsewhere} \end{cases}$$

for all $i = 1,\dots, n+1$ *where*

$$A_i = A\begin{pmatrix} u_1 & \cdots & u_{i-1} u_{i+1} & \cdots & u_{n+1} \\ M_{-m+1} & & \cdots & & M_{n-m} \end{pmatrix}.$$

(c) *Let* R_1 *be a reference. Then there exist unique integers* p *and* q *satisfying* (2.3).

Proof. (a) We see that $S_m(\tilde{I}, R) = S_m(J_R, R)$. It follows from (2.1) and (2.4) that

$$\det A \begin{pmatrix} u_p & \cdots & u_{i-1} u_{i+1} & \cdots & u_{q+1} \\ M_{-m+p} & & \cdots & & M_{-m+q} \end{pmatrix} \neq 0$$

for all $i = p, \ldots, q+1$. Hence $S_m(\tilde{I}, R)$ satisfies the Haar condition.

(b) We define R_1 such that $u_i \in (x_{-m+i}, x_i)$, $i = 1, \ldots, p-1$, and $u_{i+1} \in (x_{-m+i}, x_i)$, $i = q, \ldots, n$. It is obvious that $R_1 = \{u_i\}_{i=1}^{n+1}$ is a reference.

Let $V_i = \{u_1, \ldots, u_{n+1}\} \backslash \{u_i\} = \{v_i\}_{i=1}^{n}$. Suppose that $u_i \in R$. Then it follows from (2.1) and (2.4) that $\det(A_i) \neq 0$. Suppose that $u_i \notin R$. Let $i < p$ then we conclude from (2.4) that (x_{-m+1}, x_{p-1}) contains only the points $u_1, \ldots, u_{p-1}$. Hence $\{u_1, \ldots, u_{p-1}\} \backslash \{u_i\} = \{v_1, \ldots, v_{p-2}\}$ and $v_{p-1} \geqslant x_{p-1}$. The conditions $v_i \in (x_{-m+i}, x_i)$, $i = 1, \ldots, p-1$, cannot be satisfied and the rank of A_i is less than n. Similarly the case $i > q+1$ can be proven.

(c) Let (p_1, q_1) and (p_2, q_2) be integers satisfying (2.3). Then it follows from the above arguments that $\det(A_i) \neq 0$, iff $i = p_1, \ldots, q_1 + 1$ and $\det(A_i) \neq 0$, iff $i = p_2, \ldots, q_2 + 1$. This contradiction completes the proof.

Now we are able to give characterization theorems for our approximation problem.

THEOREM 2.4. *Let the partition* $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$ *be given and let* $\tilde{I} = (x_{-m+1}, x_n)$. *Let* T *be a compact subset of* $\tilde{I}$ *such that* $\dim(S_m(\tilde{I}, T)) = n$.

(a) *Then* s_0 *in* $S_m(\tilde{I}, T)$ *is a best approximation to a function* f *in* $C(T)$ *out of* $S_m(\tilde{I}, T)$ *if and only if there exists a subset* $R = \{u_i\}_{i=p}^{q+1}$ *of* T *which is associated with a subinterval* J_R *of the form* (2.4) *such that*

$$(f - s_0)(u_i) = \eta(-1)^i \, \|(f - s_0)\|, \qquad i = p, \ldots, q+1, \ \eta \in \{-1, 1\}. \quad (2.5)$$

(b) *If* s_1 *is a best approximation where* $s_0 \neq s_1$ *then*

$$s_0(x) = s_1(x) \qquad \text{for all} \quad x \in J_R.$$

Proof. (a) Let s_0 be a best approximation. It is well-known that every $s \in S_m(\tilde{I}, T)$ has at most $n-1$ sign changes, i.e., there do not exist $n+1$ points $t_1 < t_2 < \cdots < t_{n+1}$ in T with $s(t_i)\, s(t_{i+1}) < 0$, $i = 1, \ldots, n$.

Then it follows from a theorem in ([2, p. 23]) that there exists a $s_1 \in P_G(f)$, $G = S_m(\tilde{I}, T)$, such that $f - s_1$ has at least $n+1$ alternating extremal points. Hence we conclude from Lemma 2.2(a) that there exists a subset R which is associated with a subinterval J_R satisfying the condition (2.5).

For the converse we assume that s_0 is a function satisfying (2.5). Then it follows from Theorem 2.3 that there exists a reference $R_1 = \{u_i\}_{i=1}^{n+1}$ which contains R. We conclude from Section 1 that there exists a unique vector $\bar{\lambda} = (\lambda_1, ..., \lambda_{n+1})^T$ (up to a scalar) satisfying

$$\bar{\lambda}^T A \begin{pmatrix} u_1 & \cdots & u_{n+1} \\ M_{-m+1} & \cdots & M_{n-m} \end{pmatrix} = 0.$$

It is well known that

$$\det A \begin{pmatrix} t_1 & \cdots & t_n \\ M_{-m+1} & \cdots & M_{n-m} \end{pmatrix} \geqslant 0$$

for all $x_{-m+1} < t_1 < t_2 < \cdots < t_n < x_n$. Thus we obtain from (1.1) that $\lambda_i \cdot \lambda_{i+1} \leqslant 0$, $i = 1, ..., n$. It follows from Theorem 2.3(b) that $\lambda_i \neq 0$ iff $i \in \{p, ..., q+1\}$.

Therefore $\eta\lambda_i \operatorname{sign}((f - s_0)(u_i)) \geqslant 0$ for $i = 1, ..., n+1$, $\eta \in \{-1, 1\}$. Hence $\eta\bar{\lambda}$ is a vector satisfying the conditions of Theorem 1.1 and s_0 is a best approximation to f on R_1. It is obvious that s_0 is also a best approximation on T. This completes the proof of (a).

(b) It follows from (a) that

$$\max_{x \in R} |f(x) - s_1(x)| \leqslant \max_{x \in R} |f(x) - s_0(x)|.$$

We conclude from Theorem 2.3(a) that $S_m(\tilde{I}, R)$ satisfies the Haar condition. Hence $s_0(x) = s_1(x)$ for all $x \in R$ and it follows from (2.1) and (2.4) that $(s_0 - s_1)(x) = 0$ for $x \in J_R$.

It will be necessary to establish characterization theorems for all kinds of boundary conditions. Therefore we have to define subsets associated with subintervals of the form (2.4) for all classes of subspaces.

DEFINITION 2.5. Let the partition $\Delta = \{x_i\}_{i=0}^{k+1}$ be given and let I be an interval such that $(x_0, x_{k+1}) \subset I \subset [x_0, x_{k+1}]$. We call a subset $R = \{u_i\}_{i=p}^{q+1}$ *associated with a subinterval* J_R relative to $S_m(I)$, iff R and J_R, $R \subset J_R$, satisfy the following conditions:

(a) Let $I \in K_1$, i.e., $S_m(I) = \operatorname{span}\{1, ..., x^{m-1}, (x - x_1)_+^{m-1}, ..., (x - x_k)_+^{m-1}\}$: $J_R = [x_{p-1}, x_{-m+q+1}]$ and $u_i \in (x_{-m+i}, x_{i-1})$, $i = p+1, ..., q$.

(b) Let $I \in K_2$, i.e., $S_m(I) = \operatorname{span}\{(x - x_0)_+^{m-1}, ..., (x - x_k)_+^{m-1}\}$: $J_R = (x_0, x_q]$ if $p = 1$, $J_R = [x_{p+m-2}, x_q]$ if $p > 1$ and $u_i \in (x_{i-1}, x_{i+m-2})$, $i = p+1, ..., q$.

(c) Let $I \in K_3$, i.e., $S_m(I) = \text{span}\{(x_1 - x)_+^{m-1}, \ldots, (x_{k+1} - x)_+^{m-1}\}$: $J_R = [x_{p-1}, x_{k+1})$ if $q = k+1$, $J_R = [x_{p-1}, x_{-m+q+1}]$ if $q < k+1$ and $u_i \in (x_{-m+i}, x_{i-1})$, $i = p+1, \ldots, q$.

(d) Let $I \in K_4$ and $k \geqslant m-1$, i.e., $S_m(I) = \text{span}\{M_0, \ldots, M_{k-m+1}\}$: $J_R = (x_0, x_{k+1})$ if $p = 1$, $q = k - m + 2$; $J_R = [x_{p+m-2}, x_{k+1})$ if $p > 1$, $q = k - m + 2$; $J_R = (x_0, x_q]$ if $p = 1$, $q < k - m + 2$; $J_R = [x_{p+m-2}, x_q]$ if $p > 1$, $q < k - m + 2$ and $u_i \in (x_{i-1}, x_{i+m-2})$, $i = p+1, \ldots, q$.

Then we obtain the following result:

THEOREM 2.6. *Let the partition* $\Delta = \{x_i\}_{i=0}^{k+1}$ *be given, let* I *be a subinterval such that* $(x_0, x_{k+1}) \subset I \subset [x_0, x_{k+1}]$ *and let* T *be a compact subset of* I *satisfying* $\dim(S_m(I, T)) = \dim(S_m(I))$.

(a) *Then a function* s_0 *is a best approximation to a function* f *in* $C(T)$ *out of* $S_m(I, T)$ *if and only if there exists a subset* $R = \{u_i\}_{i=p}^{q+1}$ *which is associated with a subinterval* J_R *such that*

$$(f - s_0)(u_i) = \eta(-1)^i \,\|(f - s_0)\|, \qquad i = p, \ldots, q+1, \ \eta \in \{-1, 1\}. \quad (2.6)$$

(b) *If* s_1 *is a best approximation where* $s_0 \neq s_1$ *then* $s_0(x) = s_1(x)$ *for all* $x \in J_R$.

Proof. Theorem 2.4 and Remark 2.1.

For $S_m(I, T) = S_m(\Delta)$ the statements of Theorem 2.6 are due to Rice [3, pp. 151, 152] and Schumaker [4].

3. CHARACTERIZATION THEOREMS FOR STRICT APPROXIMATIONS

The best approximations of Section 2 are not uniquely determined in general. Therefore it is natural to consider conditions which single out one of the best approximations. Descloux [1] and Rice [3] considered the so-called strict approximation. First we shall state the definition of these best approximations (see [3]). Let f be a function of $C(T)$, where T is a finite subset of $\mathbb{R}$ and let G be an n-dimensional subspace of $C(T)$. Suppose g_0 is a best approximation from G to f on T. A subset S of the extremal points of $f - g_0$ is said to be a *critical point set* if g_0 is a best approximation to f on S but is not a best approximation to f on any proper subset of S.

It follows from Theorem 1.1 that a critical point set contains at most $n + 1$ points.

Now we are able to define strict approximations.

DEFINITION 3.1. Let a finite subset T of $\mathbb{R}$ be given. Let f be a function in $C(T)$ and G be an n-dimensional subspace G of $C(T)$. Set $G_0 = G$ and $T_0 = \varnothing$. Then we define for $j \geqslant 1$

$$G_j = \{\tilde{g} \in G_{j-1}: \max_x |(f - \tilde{g})(x)| \leqslant \max_x |(f - g)(x)| \text{ for all } x \in T \backslash T_{j-1} \text{ and all } g \in G_{j-1}\}.$$

Denote by H_j the set of critical point sets with respect to a function $g_j \in G_j$ and denote by V_j the set of points which are contained in a critical point set of H_j. Set $T_j = T_{j-1} \cup V_j$. The construction is continued until $T = T_l$ for some l. The members of G_l are said to be *strict approximations* to f on T.

The strict approximation is unique (see [3, p. 243]).

Next we shall determine the critical point sets of our approximation problems in Section 2.

THEOREM 3.2. *Let the assumptions of Theorem* 2.6 *be given and let* s_0 *be a best approximation from* $S_m(I, T)$ *to a function* f *in* $C(T)$. *Then the following conditions are equivalent*:

(a) *The subset* $R \subset T$ *is a critical point set.*

(b) *The subset* $R \subset T$ *is associated with a subinterval* J_R *relative to* $S_m(I)$ *and* $f - s_0$ *has alternating extremal points on* R.

Proof. (a) → (b). It follows from the definition of critical point sets that s_0 is a best approximation to the function f on R. Hence we conclude from Theorem 2.6 that there exists a subset $R_1 \subset R$ associated with a subinterval such that $f - s_0$ has alternating extremal points on R_1. If R_1 is a proper subset of R then s_0 is also a best approximation on R_1. This contradiction completes the proof.

(b) → (a). We conclude from Theorem 2.6 that $s_0|_R$ is a best approximation from $S_m(J_R, R)$ to $f|_R$. Let t be a point of R and $R_2 = R \backslash \{t\} = \{v_i\}_{i=p}^{q}$. Then it follows from the assumptions on R that $v_i \in (x_{-m+i}, x_i)$, $i = p, \ldots, q$, and from (2.1) that $f|_{R_2} \in S_m(J_R, R_2)$. Hence s_0 is not a best approximation to f on R_2. This proves the theorem.

Remark 3.3. It follows from Theorem 2.3(c) that a best approximation on a reference has a unique critical point set.

DEFINITION 3.4. Let the assumptions of Theorem 2.6 be given and let s_0 be a best approximation from $S_m(I, T)$ to f in $C(T)$. Suppose R is a critical point set. Then there exists a subinterval J_R which is associated with R. We call this subinterval J_R *associated with the critical point set* R. Let s_1 be a best approximation to f on a reference $U \subset T$ and let J_U be the subinterval associated with the unique critical point set of U. Then we call the subinterval J_U to be *associated with the reference.*

It follows immediately from Theorem 2.6 and Theorem 3.2 that $s_0|_{J_R} = s_1|_{J_R}$ if s_0 and s_1 are two best approximations and J_R is a subinterval associated with a critical point set relative to s_0. Hence the best approximations are uniquely determined on subintervals which are associated with critical point sets. Using this property we shall see that it is possible to define strict approximations otherwise than in Definition 3.1.

First we shall give the following notation:

Let the assumptions of Theorem 2.6 be given and let s_0 be a best approximation from $S_m(I, T)$ to f in $C(T)$. Suppose that H is the set of all subintervals I_{p_i,q_i} associated with a critical point set of $f - s_0$. Then we denote by I_H the following set:

$$I_H = \{x\text{: there exists a subinterval } I_{p_i,q_i} \in H \text{ such that } x \in I_{p_i,q_i}\}$$
$$\cup \{x\text{: there exist subintervals } I_{p_i,q_i}, I_{p_j,q_j} \subset H \text{ such that}$$
$$0 < p_j - q_i \leqslant m - 1 \text{ and } x \in (x_{q_i}, x_{p_j})\}.$$

We want to give some examples of critical point sets.

EXAMPLE 3.5. (a) Let the subspace $G = \operatorname{span}\{1, x, (x-1)_+, (x-2)_+\}$ be defined on the interval $I = [0, 3]$ and let the subset $T = \{1/3, 1/2, 2/3, 3/2, 7/3, 5/2, 8/3\}$ be given. Suppose that f is a function in $C(T)$ satisfying $f(1/3) = f(2/3) = f(5/2) = 1$, $f(1/2) = f(7/3) = f(8/3) = -1$ and $f(3/2) = 0$. It is obvious that 0 is the unique best approximation. The subsets $R_1 = \{1/3, 1/2, 2/3\}$ and $R_2 = \{7/3, 5/2, 8/3\}$ are critical point sets associated with the subintervals $I_1 = [0, 1]$ and $I_2 = [2, 3]$, respectively. The interval I is not associated with a critical point set. On the other hand, I_H is the interval $[0, 3]$.

(b) Suppose that the subintervals I_{p_1,q_1} and I_{p_2,q_2} satisfy $x_{p_1} < x_{p_2} \leqslant x_{q_1} < x_{q_2}$ and I_{p_1,q_1}, I_{p_2,q_2} are subintervals associated with critical point sets. We shall show by an example that the subinterval $I_{p_1,q_1} \cup I_{p_2,q_2}$ is not associated with a critical point set in general. Let the subspace $G = \operatorname{span}\{1, x, (x-1)_+, (x-2)_+, (x-3)_+\}$ be defined on the interval $I = [0, 4]$ and let the subset $T = \{1/3, 2/3, 4/3, 5/3, 5/2, 10/3, 11/3\}$ of I be given. Suppose f is a function on T such that

$$f(1/3) = f(4/3) = f(5/2) = f(11/3) = 1,$$
$$f(2/3) = f(5/3) = f(10/3) = -1.$$

It follows from Theorem 2.6 that 0 is a best approximation from G to f on T. The subintervals $I_1 = [0, 2]$ and $I_2 = [1, 4]$ are subintervals associated with critical point sets. But it can be seen that I is not an interval associated with a critical point set.

For our investigations it is necessary to show some properties of critical point sets. We shall always consider the following approximation problem:

PROBLEM AI. Let the partition $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$ be given, let $\tilde{I} = (x_{-m+1}, x_n)$ and T be a compact subset of $\tilde{I}$ such that $\dim(S_m(\tilde{I}, T)) = n$. We denote by s_0 a best approximation to f in $C(T)$ out of $G = S_m(\tilde{I}, T)$.

According to Remark 2.1 the other cases follow immediately from this approximation problem.

LEMMA 3.6. *Let Problem* AI *be given and let H be the set of all subintervals associated with a critical point set of $f - s_0$. Then I_H has the form*

$$I_H = \bigcup_{i=1}^{t} I_{p_i, q_i}, \tag{3.1}$$

$$x_{-m+1} = x_{p_0} \leqslant x_{p_1} < x_{q_1} < \cdots < x_{p_t} < x_{q_t} \leqslant x_{p_{t+1}} = x_n$$

and satisfies the conditions

(a) $p_1 = -m + 1$ *or* $1 \leqslant p_1$, $q_t = n$ *or* $q_t \leqslant n - m$

$$\begin{aligned}
I_{p_1, q_1} &= (x_{-m+1}, x_{q_1}], && \textit{if} \quad p_1 = -m + 1,\ q_1 \leqslant n - m,\\
I_{p_t, q_t} &= [x_{p_t}, x_n), && \textit{if} \quad p_t \geqslant 1,\ q_t = n,\\
I_{p_1, q_1} &= (x_{-m+1}, x_n), && \textit{if} \quad p_1 = -m + 1,\ q_t = n,\\
I_{p_i, q_i} &= [x_{p_i}, x_{q_i}] \subset [x_1, x_{n-m}] && \textit{elsewhere.}
\end{aligned}$$

(b) $p_{i+1} - q_i \geqslant m$ *for all* $i = 1, \ldots, t - 1$.

Proof. (a) Theorem 2.4 and Theorem 3.2.

(b) This condition follows from the definition of I_H.

In the next lemma we shall state an important property of the subset I_H.

LEMMA 3.7. *The best approximations are uniquely determined on the subset I_H.*

Proof. If I_0 is a subinterval associated with a critical point set then it follows from Theorem 2.6(b) that the best approximations are uniquely determined on I_0. Now let I_{p_i, q_i} and $I_{p_{i+1}, q_{i+1}}$ be subintervals satisfying $0 < p_{i+1} - q_i \leqslant m - 1$ then a spline function s on $(x_{q_i}, x_{p_{i+1}})$ is uniquely determined by $s|_{I_{p_i, q_i}}$ and $s|_{I_{p_{i+1}, q_{i+1}}}$. This completes the proof.

This lemma enables us to give another inductive definition of strict approximations.

DEFINITION 3.8. Let the partition $\Delta = \{x_i\}_{i=0}^{k+1}$ be given, let $I = [x_0, x_{k+1}]$ and $S_m(I) = \text{span}\{M_{-m+1},..., M_k\}$, where $\{M_i\}$ is the local basis defined in Section 2. Suppose that T is a finite subset of I.

Set $\bar{G}_0 = S_m(I, T)$, $H_0 = \varnothing$ and let Z_0 be the set of integers $\{-m+1,..., k\}$. Then we define for $j \geqslant 1$ the following sequence of functions s_j:

Let $\bar{G}_j$ be the set of best approximations to the function $f - (s_1 + \cdots + s_{j-1})$ on $T \cap \{I \backslash I_{H_{j-1}}\}$ out of $\text{span}\{\{M_i\}_{i \in Z_{j-1}}\}$ and let s_j be a function in $\bar{G}_j$. We denote by $\bar{H}_j$ the set of all subintervals in $I \backslash I_{H_{j-1}}$ which are associated with a critical point set of $f - (s_1 + \cdots + s_j)$. Then we define $H_j = H_{j-1} \cup \bar{H}_j$ and

$$Z_j = \{i \in Z_{j-1}: \{x: M_i(x) \neq 0\} \cap I_{H_j} = \varnothing\}.$$

The construction is continued until $Z_r = \varnothing$ for some r. Then we denote by $g(f)$ the function $s_1 + \cdots + s_r$.

We shall see that this inductive definition determines the strict approximation.

THEOREM 3.9. *Let the partition* $\Delta = \{x_i\}_{i=0}^{k+1}$ *be given, let* $I = [x_0, x_{k+1}]$ *and* $S_m(I, T)$, *where* T *is a finite subset of* I. *Then the function* $g(f)$ *defined in Definition* 3.8 *is the strict approximation to* f *out of* $S_m(I, T)$.

Proof. We shall use the notations of Definition 3.1 and Definition 3.8.

Let $G_1,..., G_l$ and $\bar{G}_1,..., \bar{G}_r$ be the sets of best approximations in these definitions. It follows immediately that $G_1 = \bar{G}_1$. Lemma 3.7 implies that the set of best approximations is uniquely determined on the subinterval I_H. Therefore we restrict ourselves in the next approximation problem in Definition 3.8 to the subset $T \cap \{I \backslash I_{H_1}\}$. In Definition 3.1 the next approximation problem is defined on $T \backslash V_1$, where V_1 is the set of points which are contained in a critical point set of H_1. It can be seen that there exists an index ν_2, $\nu_2 > 1$, such that $G_{\nu_2} = \{s_1 + s: s \in \bar{G}_2\}$. Using these arguments we can show that there exists integers ν_i such that $G_{\nu_i} = \{s_1 + \cdots + s_{i-1} + s: s \in \bar{G}_i\}$, $i = 1,..., r$. The set $\bar{G}_r$ contains exactly one function. Therefore $G_{\nu_r} = G_l = \{g(f)\}$. This completes the proof.

Now we want to state the main result of this section. In order to characterize the strict approximation g_0 of a function f we define a partition $\{I_i\}_i$ of I and consider on each subinterval I_i the best approximations to $f - g_0$ out of $S_m(I_i)$. For example, let $I_i = [x_{\nu_{i-1}}, x_{\nu_i}]$, $I_{i+1} = (x_{\nu_i}, x_{\nu_{i+1}}]$. Then $S_m(I_i)$ has no boundary conditions in x_{ν_i} while the functions of $S_m(I_{i+1})$ have a zero of order $m-1$ in x_{ν_i}. This notation will be very important to the following theorem.

THEOREM 3.10. *Let the partition* $\Delta = \{x_i\}_{i=1}^{k+1}$ *be given, let* $I = [x_0, x_{k+1}]$

and $S_m(I, T)$, *where* T *is a finite subset of* I *such that* $\dim S(I, T) = m + k$. *Suppose that* f *is a function in* $C(T)$ *and* g_0 *in* $S_m(I, T)$. *Then the following properties are equivalent*:

(a) *The function* g_0 *is the strict approximation to* f *out of* $S_m(I, T)$.

(b) *There exists a partition of the interval* $x_0 = x_{v_0} < x_{v_1} < \cdots < x_{v_h} < x_{v_{h+1}} = x_{k+1}$ *such that the subintervals* $I_i = I_{v_{i-1}, v_i}$ *satisfy the conditions*

(i) $I = \bigcup_{i=1}^{h+1} I_i$, $I_i \cap I_{i+1} = \varnothing$ *for all* $i = 1,..., h$.

(ii) 0 *is the unique best approximation from* $S_m(I_i)$ *to* $(f - g_0)$ *on* T_i, *where* $T_i = T \cap I_i$ *for all* $i = 1,..., h + 1$ *and there exists a critical point set* R_i *associated with* I_i *relative to* $S_m(I_i)$.

(iii) *Let* $\gamma_i = \max_{x \in T_i} |(f - g_0)(x)|$.

Then for all $i = 1,..., h$ *the following conditions will hold*: *If* $x_{v_i} \in I_i$ *then* $\gamma_i \geqslant \gamma_{i+1}$ *and if* $x_{v_i} \notin I_i$ *then* $\gamma_i \leqslant \gamma_{i+1}$.

Remark. The partition is not unique in general. We shall consider the problems of Example 3.5.

In (a) there exist the partitions $\{[0, 1], (1, 3]\}$ and $\{[0, 2), [2, 3]\}$.

In (b) there exist the partitions $\{[0, 2], (2, 4]\}$ and $\{[0, 1), [1, 4]\}$.

We want to illustrate Theorem 3.10 by another example: Let the following partition of the interval $I = [x_0, x_{k+1}] = [x_{v_0}, x_{v_4}]$ be given: $I_1 = [x_{v_0}, x_{v_1}]$, $I_2 = (x_{v_1}, x_{v_2}]$, $I_3 = (x_{v_2}, x_{v_3})$, $I_4 = [x_{v_3}, x_{v_4}]$. Suppose that $\{(I_i, \gamma_i)\}_{i=1}^{4}$ corresponds to the function g_0 and satisfies the properties of Theorem 3.10(b).

Then g_0 is the best approximation to f on $T \cap I_i$ out of $S_m(I) = \operatorname{span}\{1,..., x^{m-1}, (x - x_1)_+^{m-1},..., (x - x_k)_+^{m-1}\}$ for $i = 1$ and $i = 4$. 0 is the best approximation to $f - g_0$ on $T \cap I_2$ out of $S_m(I_2) = \operatorname{span}\{(x - x_{v_1})_+^{m-1},..., (x - x_{v_2-1})_+^{m-1}\}$ and 0 is the best approximation to $f - g_0$ on $T \cap I_3$ out of $S_m(I_3) = \operatorname{span}\{M_{v_2},..., M_{v_3-m}\}$, where M_i are B-splines of order m. Moreover, it follows from (b) (iii) that $\gamma_1 \geqslant \gamma_2 \geqslant \gamma_3 \leqslant \gamma_4$, where $\gamma_i = \|(f - g_0)|_{T \cap I_i}\|$.

For the proof of Theorem 3.10 it is necessary to show some further properties of critical point sets. We shall prove two lemmas and always consider the Problem AI. The other cases can be similarly handled.

LEMMA 3.11. *Let Problem* AI *be given and let* H *be the set of all subintervals associated with a critical point set of* $f - s_0$. *Let* I_H *be of the form of Lemma* 3.6

$$I_H = \bigcup_{i=1}^{t} I_{p_i, q_i}, \qquad x_{-m+1} \leqslant x_{p_1} < x_{q_1} < \cdots < x_{p_t} < x_{q_t} \leqslant x_n.$$

Then for each subinterval I_{p_i,q_i} *there exists a partition* $\Delta_i: x_{p_i} = x_{p_i+v_0} < x_{p_i+v_1} < \cdots < x_{p_i+v_{i-1}} < x_{p_i+v_i} = x_{q_i}$ *such that for*

$$\tilde{I}_j = I_{p_i+v_{j-1}, p_i+v_j}, \qquad \tilde{T}_j = T \cap \tilde{I}_j, \qquad j = 1,..., i$$

the following conditions will hold:

(a) $\bigcup_{j=1}^{i} \tilde{I}_j = I_{p_i,q_i}$, $\tilde{I}_j \cap \tilde{I}_{j+1} = \varnothing$ *for* $j = 1,..., i-1$.

(b) 0 *is a best approximation from* $S_m(\tilde{I})$ *to* $f - s_0$ *on* $\tilde{T}_j$ *and* $\tilde{I}_j$ *is associated with a critical point set of* $f - s_0$.

Proof. Let I_{p_i,q_i} be given. Suppose that I_{μ_1,ν_1} and I_{μ_2,ν_2} are subintervals of I_{p_i,q_i} which are associated with critical point sets satisfying $x_{\mu_1} < x_{\mu_2}$, $x_{\nu_1} < x_{\nu_2}$ and $\mu_2 - \nu_1 < m$.

We only consider the case $I_{p_i,q_i} = [x_{p_i}, x_{q_i}]$, $I_{\mu_1,\nu_1} = [x_{\mu_1}, x_{\nu_1}]$ and $I_{\mu_2,\nu_2} = [x_{\mu_2}, x_{\nu_2}]$. The other cases can be similarly shown.

It follows from Theorem 2.6 that there exist points $\{v_i\}_{i=\mu_2+1}^{m+\nu_2} \subset \{T \cap I_{\mu_2,\nu_2}\}$ such that $f - s_0$ has alternating extremal points on $\{v_i\}_{i=\mu_2+1}^{m+\nu_2}$ and

$$v_i \in (x_{-m+i}, x_{i-1}), \qquad i = \mu_2 + 2,..., m + \nu_2 - 1. \tag{3.2}$$

Then we conclude from (3.2) that $V = \{v_i\}_{i=m+\nu_1}^{m+\nu_2} \subset (x_{\nu_1}, x_{\nu_2}]$ and

$$v_i \in (x_{-m+i}, x_{i-1}), \qquad i = m + \nu_1 + 1,..., m + \nu_2 - 1.$$

Hence it follows from Theorem 2.6 that 0 is a best approximation from $S_m(I_2)$ to $f - s_0$ on T_2, where $I_2 = (x_{\nu_1}, x_{\nu_2}]$ and $T_2 = T \cap I_2$. Moreover I_2 is associated with the critical point set V relative to $S_m(I_2)$.

It follows from the definition of I_H that there exist subintervals $I_{\mu_j,\nu_j} = [x_{\mu_j}, x_{\nu_j}]$ of I_{p_i,q_i} associated with critical point sets, $j = 1,..., l$, such that $x_{\mu_j} < x_{\mu_{j+1}}$, $x_{\nu_j} < x_{\nu_{j+1}}$, $\mu_{j+1} - \nu_j < m$ and $x_{\mu_1} = x_{p_i}$, $x_{\nu_l} = x_{q_i}$. We see that a repeated application of the above arguments yields a partition satisfying the properties of the theorem on I_{p_i,q_i}. Hence we obtain a partition for all subintervals I_{p_i,q_i}. This completes the proof.

LEMMA 3.12. *Let Problem* AI *be given and let* $I_1 = I_{p_1,q_1}$ *and* $I_2 = I_{q_1,q_2}$, *where* $x_{p_1} < x_{q_1} < x_{q_2}$, $x_{q_1} \in I_1$ *and* $x_{q_1} \notin I_2$ *be two subintervals associated with critical point sets of* $f - s_0$ *relative to* $S_m(I_1)$ *and* $S_m(I_2)$, *respectively. Then there exists a subinterval* $I_0 = I_{p_2,q_2}$ *where*

$$x_{p_1} \leqslant x_{p_2} < x_{q_2}, \qquad p_2 - q_1 < m$$

$$x_{p_2} \in I_{p_2,q_2} \quad \textit{if } p_2 > -m+1, \qquad x_{q_2} \in I_{p_2,q_2} \quad \textit{if } x_{q_2} \in I_2$$

such that I_0 *is associated with a critical point set* R_0 *relative to* $S_m(I_0)$.

Proof. We only consider the case where $p_1 \geqslant 1$, $q_2 \leqslant k$ and $n = m + k$. Hence $I_1 = [x_{p_1}, x_{q_1}]$ and $I_2 = (x_{q_1}, x_{q_2}]$. The other cases can be similarly shown. We conclude from Theorem 2.6 that there exist sets $\{u_i\}_{i=p_1+1}^{m+q_1} \subset T_1$ and $\{v_i\}_{i=m+q_1}^{m+q_2} \subset T_2$, $T_i = T \cap I_i$ for $i = 1, 2$, such that

$$\|(f - s_0)|_{T_1}\| = \xi(-1)^i (f - s_1)(u_i), \qquad i = p_1 + 1, \ldots, m + q_1, \quad \xi^2 = 1$$

$$\|(f - s_0)|_{T_2}\| = \eta(-1)^i (f - s_1)(v_i), \qquad i = m + q_1, \ldots, m + q_2, \quad \eta^2 = 1$$

and

$$\begin{aligned} u_i &\in (x_{-m+i}, x_{i-1}), \qquad i = p_1 + 2, \ldots, m + q_1 - 1, \\ v_i &\in (x_{-m+i}, x_{i-1}), \qquad i = m + q_1 + 1, \ldots, m + q_2 - 1. \end{aligned} \tag{3.3}$$

We define a subset $Y_1 = \{y_i\}_{i=p_1+1}^{m+q_2}$ in the following way

$$y_i = \begin{cases} u_i & \text{if} \quad (f - s_1)(u_{m+q_1})(f - s_1)(v_{m+q_1}) > 0, \\ u_{i+1} & \text{elsewhere} \end{cases}$$

for all

$$i = p_1 + 1, \ldots, m + q_1 - 1$$

and

$$y_i = v_j, \qquad i = m + q_1, \ldots, m + q_2.$$

It follows that

$$(f - s_1)(y_i)(f - s_1)(y_{i+1}) < 0, \qquad i = p_1 + 1, \ldots, m + q_2 - 1. \tag{3.4}$$

We have to distinguish the following cases:

(1) If $x_{m+q_1-1} \leqslant v_{m+q_1}$ then $I_0 = [x_{m+q_1-1}, x_{q_2}]$ and $R_0 = \{v_i\}_{i=m+q_1}^{m+q_2}$ satisfy the conditions of the lemma.

(2) Let $v_{m+q_1} < x_{m+q_1-1}$ and let $c := (f - s_1)(u_{m+q_1}) \times (f - s_1)(v_{m+q_1}) > 0$. Then it follows from (3.3) and $x_{q_1} < v_{m+q_1} < x_{m+q_1-1}$ that

$$y_i \in (x_{-m+i}, x_{i-1}), \qquad i = p_1 + 2, \ldots, m + q_2 - 1$$

and we set

$$I_0 = [x_{p_1}, x_{q_2}] \qquad \text{and} \qquad R_0 = Y_1.$$

(3) Let $v_{m+q_1} < x_{m+q_1-1}$ and $c < 0$. Then it follows from (3.3) and $u_{m+q_1} \in (x_{q_1-1}, x_{q_1}]$ that

$$y_i \in (x_{-m+i}, x_{i-1}), \qquad i = m+q_1,..., m+q_2-1,$$
$$y_i \in (x_{-m+i}, x_i), \qquad i = p_1+1,..., m+q_1-1. \tag{3.5}$$

Let p_2 be the integer in $p_1 \leqslant p_2 \leqslant q_1$ such that

$$y_{p_2+1} \geqslant x_{p_2}, \qquad y_i < x_{i-1}, \qquad i = p_2+2,..., m+q_1-1. \tag{3.6}$$

Then we conclude from (3.5) and (3.6) that

$$y_i \in (x_{-m+i}, x_{i-1}), \qquad i = p_2+2,..., m+q_2-1.$$

Set

$$I_0 = [x_{p_2}, x_{q_2}], \qquad R_0 = \{y_i\}_{i=p_2+1}^{m+q_2}.$$

R_0 is a reference relative to $S_m(I_0)$ and I_0 is associated with this reference.

According to (3.4), $f - s_0$ alternates on R_0. Therefore s_0 is a best approximation from $S_m(I_0)$ to f on $T \cap I_0$. Hence R_0 is a critical point set associated with I_0. This completes the proof.

Proof of Theorem 3.7. (a) → (b). First we shall assume that g_0 is the strict approximation. According to Theorem 3.9 the strict approximation can be constructed by Definition 3.8. We shall use the notations of this definition. Set $h_j = \sum_{i=1}^{j} s_j$, $j = 1,..., r$, then $h_r = g_0$ is the strict approximation.

Let $I_{H_1} = I_{\bar{H}_1} = \bigcup_{i=1}^{t} I_{p_i, q_i}$ be associated with the critical point sets of $f - h_1$. It follows from Lemma 3.6 that $q_{i+1} - p_i \geqslant m$ for all $i = 1,..., t-1$. We conclude from Definition 3.8 that $(f - h_1)|_{T \cap I_{\bar{H}_1}} = (f - h_r)|_{T \cap I_{\bar{H}_1}}$. Lemma 3.11 implies that there exists a partition on $I_{\bar{H}_1}$ satisfying the properties of the theorem. Next we consider the subintervals $(x_{q_i}, x_{p_{i+1}})$, $i = 1,..., t-1$, $[x_0, x_{p_1})$ and $(x_{q_t}, x_{k+1}]$. We have $I_{\bar{H}_2} \subset I \backslash I_{\bar{H}_1}$, where $\bar{H}_2$ is the set of subintervals associated with critical point sets of $f - h_2$ relative to $S_m(I \backslash I_{H_1})$. Moreover, $(f - h_2)|_{T \cap I_{\bar{H}_2}} = (f - h_r)|_{T \cap I_{\bar{H}_2}}$. Using Lemma 3.11 we obtain a partition of $I_{\bar{H}_2}$. This construction is continued until $I = I_{H_r}$. We obtain a partition of I satisfying properties (i) and (ii) of the theorem.

Let $I_i = I_{v_i-1, v_i}$ and $I_{i+1} = I_{v_i, v_{i+1}}$ be two subintervals of this partition. Assume that $x_{v_i} \in I_i$ and $x_{v_i} \notin I_{i+1}$. Then it follows from the construction of the strict approximation that $I_i \in \bar{H}_{\eta_1}$ and $I_{i+1} \in \bar{H}_{\eta_2}$ satisfy $\eta_1 \leqslant \eta_2$. Therefore $\|(f - h_r)|_{I_i}\| \geqslant \|(f - h_r)|_{I_{i+1}}\|$. The case $x_{v_i} \notin I_i$ and $x_{v_i} \in I_{i+1}$ can be similarly handled. This proves (iii).

(b) → (a). For the converse let a function g_0 in $S_m(I, T)$ be given such that there exists a partition $\{(I_i, \gamma_i)\}_{i=1}^{h+1}$ of I satisfying the conditions of the theorem.

First we define the set of subintervals $C_1 = \{I_i: \gamma_i = \delta_1\}$, where $\delta_1 = \max_{i=1,\ldots,h+1} \gamma_i$. We shall show that $I_{C_1} = I_{\bar{H}_1}$. Let $I_i = I_{\nu_{i-1},\nu_i}$ be an element of C_1. Then we prove that $I_i \subset I_{\bar{H}_1}$. We must distinguish the following cases:

(α) $I_i \in K_1$, i.e., $I_i = [x_{\nu_{i-1}}, x_{\nu_i}]$. Then it is obvious that $I_i \in \bar{H}_1$. Hence $I_i \subset I_{\bar{H}_1}$.

(β) $I_i \in K_2$, i.e., $I_i = (x_{\nu_{i-1}}, x_{\nu_i}]$. Then it follows from property (b)(iii) that there exist subintervals $\{I_j\}_{j=i-\mu}^{i}$ satisfying $I_j \in C_1$, $j = i-\mu,\ldots, i$ and $I_{i-\mu} \in K_1$, $I_{i-\mu+1},\ldots, I_{i-1} \subset K_2$. We conclude from Lemma 3.12 that there exists an interval $\tilde{I} = [x_{\eta_1}, x_{\nu_{i-\mu+1}}] \subset I_{i-\mu} \cup I_{i-\mu+1}$, where $x_{\nu_{i-\mu-1}} \leqslant x_{\eta_1} < x_{\nu_{i-\mu-1}}$ and $\eta_1 - \nu_{i-\mu} < m$ such that $\tilde{I}$ associated with a critical point set of $f - g_0$ relative to $S_m(\tilde{I})$. Then it follows from the construction of $I_{\bar{H}_1}$ that $\{I_{i-\mu} \cup I_{i-\mu+1}\} \subset I_{\bar{H}_1}$. A repeated application of these arguments shows that $\{I_{i-\mu} \cup \cdots \cup I_i\} \subset I_{\bar{H}_1}$. $I_i \in K_3$ can be similarly handled.

(γ) $I_i \in K_4$, i.e., $I_i = (x_{\nu_{i-1}}, x_{\nu_i})$. Then it follows from the assumption (b)(iii) and the arguments in (β) that there exists a subinterval $I^1 = [x_r, x_{\nu_{i-1}}] \subset \bar{H}_1$. We apply Lemma 3.12 to I^1, I_i and obtain a subinterval $I^2 = [x_{\eta_1}, x_{\nu_i}) \in K_3$ such that I^2 is associated with a critical point set of $f - g_0$ relative to $S_m(I^2)$, where $x_r \leqslant x_{\eta_1} < x_{\nu_i}$ and $\eta_1 - \nu_{i-1} < m$. Then it follows from (β) that $I^2 \subset I_{\bar{H}_1}$. We conclude from $\eta_1 - \nu_{i-1} < m$ that $I^1 \cup I_i \subset I_{\bar{H}_1}$. Therefore $I_{C_1} \subset I_{\bar{H}_1}$. Moreover, we have $|(f - g_0)(x)| < \delta_1$ for all $x \in T \cap (I \backslash I_{\bar{H}_1})$. Thus we obtain $I_{C_1} = I_{\bar{H}_1}$. Now we consider the approximation problems corresponding to the subsets $\tilde{G}_j$ of Definition 3.8. Let $\delta_i = \max_j \{\gamma_j: \gamma_j < \delta_{i-1}\}$ for $i \geqslant 2$. Using the above arguments we are able to show that $I_{C_i} = I_{\bar{H}_i}$ and 0 is a best approximation from $S_m(I \backslash I_{H_{i-1}})$ to $f - g_0$ on $T \cap \{I \backslash I_{H_{i-1}}\}$ for $i = 2,\ldots, r$. Hence g_0 is a function satisfying the properties of Definition 3.8. Therefore, g_0 is the strict approximation.

A modification of Definition 3.8 yields a partition of the interval satisfying the properties of Theorem 3.10.

DEFINITION 3.13. Let $\Delta = \{x_i\}_{i=0}^{k+1}$ and $I = [x_0, x_{k+1}]$ be given. Let $S_m(I) = \operatorname{span}\{M_{-m+1},\ldots, M_k\}$, where $\{M_i\}$ is a local basis. Suppose that T is a finite subset of I.

Set $\tilde{G}_0 = S_m(I, T)$, $\tilde{I}_0 = \varnothing$ and let Z_0 be the set of integers $\{-m+1,\ldots, k\}$. Then we define for $j \geqslant 1$ the following sequence of functions s_j:

Let $\tilde{G}_j$ be the set of best approximations to the function $f - (s_1 + \cdots + s_{j-1})$ (i.e., f if $j = 1$) on $T \cap \{I \backslash \tilde{I}_{j-1}\}$ out of $\operatorname{span}\{\{M_i\}_{i \in Z_{j-1}}\}$ and let s_j be a function in $\tilde{G}_j$. Suppose that I_j is a subinterval of $I \backslash \tilde{I}_{j-1}$ which is associated with a critical point set of $f - (s_1 + \cdots + s_j)$. Then we define $\tilde{I}_j = \tilde{I}_{j-1} \cup I_j$ and

$$Z_j = \{i \in Z_{j-1}: \{x: M_i(x) \neq 0\} \cap \tilde{I}_j = \varnothing\}.$$

This construction is continued until $Z_t = \emptyset$ for some t. We denote by $\tilde{g}(f)$ the function $s_1 + \cdots + s_t$.

COROLLARY 3.14. *The function $\tilde{g}(f)$ of Definition* 3.13 *is the strict approximation and the subintervals I_i are a partition satisfying the properties of Theorem* 3.10.

Proof. It follows from the construction of Definition 3.13 that the partition $\{I_i\}$ satisfies the conditions of Theorem 3.10. Therefore $\tilde{g}(f)$ is the strict approximation.

4. APPROXIMATIONS ON AN INTERVAL

In this section we want to consider an approximation problem which is defined on an interval, i.e., $T = [a, b] = [x_0, x_{k+1}]$. If we apply the construction of Definition 3.13 to this case then we shall not obtain a function which is uniquely determined in general. We shall see the difficulties in the following example:

Let Problem AI of Section 3 be given, let $T = [x_{-m+1}, x_n]$ and f be a function in $C(T)$. Therefore we consider the best approximations to f out of a subspace spanned by B-splines. It is possible that there does not exist a critical point set in $\tilde{I} = (x_{-m+1}, x_n)$. Then a boundary point x_{-m+1} or x_n is a critical point set and the deviation of the best approximation is $|f(x_{-m+1})|$ or $|f(x_n)|$ because $s(x_{-m+1}) = s(x_n) = 0$ for all $s \in S_m(\tilde{I}, T)$. In this case the best approximations are uniquely determined only in one boundary point in general. Therefore we cannot apply the construction of Definition 3.13. We see that the critical point set is not contained in $\tilde{I}$.

The construction can only be applied to a subset of $C[a, b]$.

Let $\Delta = \{x_i\}_{i=0}^{k+1}$ and $T = [x_0, x_{k+1}] = [a, b]$ be given. Then $S_m(I, T) = S_m(\Delta)$.

The construction of Definition 3.13 is possible for a function $f \in C[a, b]$ if f satisfies the following conditions.

Let $\tilde{G}_j$ be the set of best approximations as in Definition 3.13. We assume that there exists a subinterval I_j of $I \backslash \tilde{I}_{j-1}$ which is associated with a critical point set of $f - (s_1 + \cdots + s_j)$ for all j. We have seen in the above example that this condition is not satisfied in general.

Then the construction of Definition 3.13 defines a unique function $\tilde{g}(f) \in S_m(\Delta)$.

We denote the set of functions f in $C[a, b]$ satisfying the above conditions by $\tilde{C}[a, b]$.

THEOREM 4.1. *Let a function f in $\tilde{C}[a, b]$ and the subspace $S_m(\Delta)$ be* *given. Then there exists a function* g_0 *in* $S_m(\Delta)$ *such that the following assertions are true*:

(a) *There exists a partition of the interval* $x_0 = x_{\nu_0} < x_{\nu_1} < \cdots < x_{\nu_h} < x_{\nu_{h+1}} = x_{k+1}$ *such that the subintervals* $I_i = I_{\nu_{i-1}, \nu_i}$ *satisfy*:

(i) $I = \bigcup_{i=1}^{h+1} I_i$, $I_i \cap I_{i+1} = \emptyset$ *for all* $i = 1,\ldots, h$.

(ii) 0 *is the unique best approximation from* $S_m(I_i)$ to $(f - g_0)$ *on* T_i, *where* $T_i = [x_{\nu_{i-1}}, x_{\nu_i}]$ *and there exists a critical point set* $R_i \subset I_i$ *which is associated with* I_i *for all* $i = 1,\ldots, h+1$.

(iii) *Let* $\gamma_i = \max_{x \in T_i} |(f - g_0)(x)|$. *Then for all* $i = 1,\ldots, h$ *the following conditions will hold*: *If* $x_{\nu_i} \in I_i$ *then* $\gamma_i \geqslant \gamma_{i+1}$ *and if* $x_{\nu_i} \in I_{i+1}$ *then* $\gamma_i \leqslant \gamma_{i+1}$.

(b) *The function* g_0 *is the strict approximation to* f *on the subset* $R = \bigcup_{i=1}^{h+1} R_i$, *where* R_i *are the critical point sets of* (a).

Proof. (a) It follows from the construction of Definition 3.13 that there exists a function g_0 satisfying (i)–(iii).

(b) The function $g_0|_R$ satisfies the conditions of Theorem 3.10. Therefore g_0 is a strict approximation for the problem defined on R.

Strict approximations are not defined for continuous problems. Since the function g_0 of Theorem 4.1 corresponding to a function $f \in \tilde{C}[a, b]$ is also a strict approximation on a finite subset of $[a, b]$ we shall call this function a strict approximation for the continuous problem.

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